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ON UNIQUENESS OF SOLUTIONS IN GENERALIZED PLASTICITY AT INFINITESIMAL DEFORMATION

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Abstract—By means of the usual methodology used in proving the uniqueness of solutions of quasi-static boundary-value problems in solid mechanics with infinitesimal deformations, it is shown that in a material modeled by generalized plasticity theory, a sufficient condition for uniqueness of the strain-rate field holds if the material is not merely work-hardening but if its plastic modulus exceeds a certain threshold value that increases with the degree of deviation of the plastic strain rate from the normal to the loading surface.

1. INTRODUCTION

Sufficient conditions for the uniqueness of the strain-rate fields in solids described by classical plasticity theory were derived by Hill[1] for materials obeying an associated flow law, according to which the plastic strain rate must be normal to the yield surface, and extended by Raniecki[2] for materials with non-associated flow laws. It is the purpose of this paper to derive analogous conditions for solids described by generalized plasticity theory[3–5], in which a yield surface is not necessary.

By generalized plasticity I mean a model of rate-independent inelastic behavior with elastic range, which at infinitesimal deformation can be described as follows. Let $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ denote an array of internal variables, so that the isothermal stress–strain relation (with the temperature dependence not shown) is $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \boldsymbol{\xi})$ and the corresponding strain–stress relation is $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{\sigma}, \boldsymbol{\xi})$. The plastic strain rate $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}$ is defined as

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \sum_{\alpha} \frac{\partial \boldsymbol{\varepsilon}}{\partial \xi_\alpha} \dot{\xi}_\alpha \tag{1}$$

the definition essentially being that of Rice[6]. If $\mathbf{C} = \partial \boldsymbol{\sigma}/\partial \boldsymbol{\varepsilon}|_{\boldsymbol{\xi}}$ is the elastic tangent modulus (usually assumed constant in the infinitesimal theory), then

$$\dot{\boldsymbol{\sigma}} = \mathbf{C}(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}});$$

the notation used here will be one in which symmetric second-rank tensors are represented as six-dimensional vectors, so that $\mathbf{C}$ is a symmetric 6×6 matrix.

The rate equations for the internal variables ξ_α are assumed to have the form

$$\dot{\xi}_\alpha = \varphi h_\alpha(\boldsymbol{\sigma}, \boldsymbol{\xi}) \tag{2}$$

where φ is a non-negative quantity. Defining

$$\mathbf{r} = \sum_{\alpha} \frac{\partial \boldsymbol{\varepsilon}}{\partial \xi_\alpha} h_\alpha$$

we may write, by combining eqns (1) and (2)

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \varphi \mathbf{r}. \tag{3}$$

Since φ is not defined uniquely, it may be chosen to equal $|\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}|$ (where we write $|\mathbf{a}| \stackrel{\text{def}}{=} \sqrt{(\mathbf{a}^{\mathrm{T}}\mathbf{a})}$). This choice makes $\mathbf{r}$ a unit vector, i.e. $|\mathbf{r}| = 1$.

In classical plasticity, as opposed to generalized plasticity, φ is positive only when the yield criterion

$$f(\boldsymbol{\sigma}, \boldsymbol{\xi}) = 0$$

is met; an explicit form for φ is obtained by means of the consistency condition.

In generalized plasticity, a form for φ is assumed, namely

$$\varphi = \frac{\langle \mathbf{n}^{\mathrm{T}}\mathbf{C}\dot{\boldsymbol{\varepsilon}} \rangle}{S} \tag{4}$$

where S is a given positive function of the state variables, not necessarily related to any yield criterion, and $\mathbf{n}$ is the unit normal to the *loading surface* (the boundary of the *elastic range* corresponding to the given state, which is not the same as the *elastic domain* at the given $\boldsymbol{\xi}$—see Ref. [5]). $\mathbf{n}$ may be interpreted either as the normal vector in a stress space in the usual sense, or in strain space if this space is regarded as a Riemannian manifold with $\mathbf{C}$ as the metric.

The flow equation accordingly becomes

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \frac{\langle \mathbf{n}^{\mathrm{T}}\mathbf{C}\dot{\boldsymbol{\varepsilon}} \rangle}{S}\mathbf{r}. \tag{5}$$

With the help of eqn (5), the stress rate may now be written in terms of the strain rate

$$\dot{\boldsymbol{\sigma}} = \mathbf{C}\left(\dot{\boldsymbol{\varepsilon}} - \frac{\langle \mathbf{n}^{\mathrm{T}}\mathbf{C}\dot{\boldsymbol{\varepsilon}} \rangle}{S}\mathbf{r}\right). \tag{6}$$

It follows from eqn (6) that

$$\mathbf{n}^{\mathrm{T}}\dot{\boldsymbol{\sigma}} = \mathbf{n}^{\mathrm{T}}\mathbf{C}\dot{\boldsymbol{\varepsilon}} \quad \text{when} \quad \mathbf{n}^{\mathrm{T}}\mathbf{C}\dot{\boldsymbol{\varepsilon}} < 0$$

and

$$\mathbf{n}^{\mathrm{T}}\dot{\boldsymbol{\sigma}} = \mathbf{n}^{\mathrm{T}}\mathbf{C}\dot{\boldsymbol{\varepsilon}}\left(1 - \frac{\mathbf{n}^{\mathrm{T}}\mathbf{C}\mathbf{r}}{S}\right) \quad \text{when} \quad \mathbf{n}^{\mathrm{T}}\mathbf{C}\dot{\boldsymbol{\varepsilon}} > 0.$$

Consequently $\mathbf{n}^{\mathrm{T}}\dot{\boldsymbol{\sigma}}$ has the same sign as $\mathbf{n}^{\mathrm{T}}\mathbf{C}\dot{\boldsymbol{\varepsilon}}$ if and only if

$$H > 0 \tag{7}$$

where

$$H \stackrel{\text{def}}{=} S - \mathbf{n}^{\mathrm{T}}\mathbf{C}\mathbf{r} \tag{8}$$

may be regarded as the plastic modulus. Equation (7) is thus the criterion for work-hardening in generalized plasticity. If it is met, then the flow equation may be written as

$$\dot{\boldsymbol{\varepsilon}}^p = \frac{\langle \mathbf{n}^T \dot{\boldsymbol{\sigma}} \rangle}{H} \mathbf{r}.$$

2. THE UNIQUENESS THEOREM

The classical condition for the uniqueness of the strain-rate field, as formulated by Hill[7], is

$$\omega \stackrel{\text{def}}{=} (\dot{\boldsymbol{\varepsilon}}^1 - \dot{\boldsymbol{\varepsilon}}^2)^T (\dot{\boldsymbol{\sigma}}^1 - \dot{\boldsymbol{\sigma}}^2) > 0 \quad \text{unless} \quad \dot{\boldsymbol{\varepsilon}}^1 = \dot{\boldsymbol{\varepsilon}}^2$$

where $\dot{\boldsymbol{\varepsilon}}^1$ and $\dot{\boldsymbol{\varepsilon}}^2$ are two strain-rate fields compatible with the kinematic boundary conditions and the $\dot{\boldsymbol{\sigma}}^\alpha$ ($\alpha = 1, 2$) are the stress-rate field associated with $\dot{\boldsymbol{\varepsilon}}^\alpha$ through eqn (6), not necessarily statically admissible. Noting that, for any real numbers a and b

$$\langle a \rangle - \langle b \rangle = \gamma(a - b)$$

for some number γ with $0 \leqslant \gamma \leqslant 1$, we obtain

$$\dot{\boldsymbol{\sigma}}^1 - \dot{\boldsymbol{\sigma}}^2 = \left(\mathbf{C} - \frac{\gamma}{S} \mathbf{C} \mathbf{r} \mathbf{n}^T \mathbf{C} \right) (\dot{\boldsymbol{\varepsilon}}^1 - \dot{\boldsymbol{\varepsilon}}^2).$$

To simplify the writing we introduce the following definitions:

$$\mathbf{u} = \mathbf{C}^{1/2} (\dot{\boldsymbol{\varepsilon}}^1 - \dot{\boldsymbol{\varepsilon}}^2), \qquad \mathbf{p} = \frac{1}{|\mathbf{C}^{1/2}\mathbf{r}|} \mathbf{C}^{1/2} \mathbf{r}, \qquad \mathbf{q} = \frac{1}{|\mathbf{C}^{1/2}\mathbf{n}|} \mathbf{C}^{1/2} \mathbf{n}$$

so that

$$\mathbf{p}^T \mathbf{p} = \mathbf{q}^T \mathbf{q} = 1, \qquad \mathbf{p}^T \mathbf{q} = c$$

where c may be regarded as the cosine of the angle of deviation from normality.

We can now write ω as

$$\omega = \mathbf{u}^T \mathbf{u} - \frac{\gamma |\mathbf{C}^{1/2}\mathbf{r}| \, |\mathbf{C}^{1/2}\mathbf{n}|}{S} \mathbf{u}^T \mathbf{p} \mathbf{u}^T \mathbf{q}.$$

With no loss of generality, we may decompose $\mathbf{u}$ as

$$\mathbf{u} = \alpha(\mathbf{p} + \mathbf{q}) + \mathbf{v}$$

where α is some real number and $\mathbf{v}$ is a vector orthogonal to $\mathbf{p} + \mathbf{q}$, that is

$$\mathbf{v}^T (\mathbf{p} + \mathbf{q}) = 0. \tag{9}$$

It is now easy to see that

$$\mathbf{u}^T \mathbf{u} = 2\alpha^2 (1 + c) + \mathbf{v}^T \mathbf{v}$$

and

$$\mathbf{u}^T\mathbf{p}\mathbf{q}^T\mathbf{u} = \alpha^2(1 + c)^2 + (\mathbf{v}^T\mathbf{p})(\mathbf{v}^T\mathbf{q});$$

consequently

$$\omega = 2\alpha^2(1 + c) + \mathbf{v}^T\mathbf{v} - \frac{\gamma|\mathbf{C}^{1/2}\mathbf{r}|\,|\mathbf{C}^{1/2}\mathbf{n}|}{S}\alpha^2(1 + c)^2 - \frac{\gamma|\mathbf{C}^{1/2}\mathbf{r}|\,|\mathbf{C}^{1/2}\mathbf{n}|}{S}(\mathbf{v}^T\mathbf{p})(\mathbf{v}^T\mathbf{q}).$$

As a result of eqn (9), however, we have $\mathbf{v}^T\mathbf{p} = -\mathbf{v}^T\mathbf{q}$, and therefore the contribution of $\mathbf{v}$ to ω is non-negative. By disregarding it, and by setting $\gamma = 1$ (the worst case, we obtain the lower bound

$$\omega \geqslant \alpha^2(1 + c)\left[2 - \frac{|\mathbf{C}^{1/2}\mathbf{r}|\,|\mathbf{C}^{1/2}\mathbf{n}|}{S}(1 + c)\right]$$

so that a sufficient condition for the uniqueness of $\dot{\varepsilon}$ is

$$S > \frac{1}{2}|\mathbf{C}^{1/2}\mathbf{r}|\,|\mathbf{C}^{1/2}\mathbf{n}|(1 + c) = \frac{1}{2}(|\mathbf{C}^{1/2}\mathbf{r}|\,|\mathbf{C}^{1/2}\mathbf{r}| + \mathbf{n}^T\mathbf{C}\mathbf{r}). \tag{10}$$

Obviously, if the body is everywhere elastic ($\mathbf{r} = 0$) then $S > 0$ is sufficient.

In view of definition (8) of the plastic modulus H, condition (10) may be rewritten as

$$H > H_0 \tag{11}$$

where

$$H_0 = \frac{1}{2}|\mathbf{C}^{1/2}\mathbf{r}|\,|\mathbf{C}^{1/2}\mathbf{n}|(1 - c) = \frac{1}{2}(|\mathbf{C}^{1/2}\mathbf{r}|\,|\mathbf{C}^{1/2}\mathbf{n}| - \mathbf{n}^T\mathbf{C}\mathbf{r}).$$

In the form (11), the condition was derived by Raniecki[2] for solids described by classical plasticity theory. H_0, which in general depends on the state, is thus seen to be a threshold value of the plastic modulus that must everywhere be exceeded by the actual plastic modulus for the uniqueness criterion to be satisfied unconditionally. Clearly, $H_0 = 0$ at all states if and only if normality is obeyed.

It must be understood that the uniqueness condition (10) or (11) is valid for boundary-value problems in which arbitrary states of stress can occur. If, as a result of symmetry or other consideration, only a limited set of stress states is expected, then the condition may be weakened, as will be seen from the following examples.

3. EXAMPLES

Some examples of uniqueness criteria in solids with non-associated flow laws were studied by Mróz[8]. Two additional examples will be considered here.

(1) We consider first a Lévy–St. Venant material, for which the Tresca loading surface is assumed in conjunction with the Lévy (or Lévy–Mises) flow rule. Here $\mathbf{r}$ is given in tensor form by

$$r_{ij} = \frac{s_{ij}}{\sqrt{(2J_2)}} \tag{12}$$

while $\mathbf{n}$ is most easily expressed tensorially with respect to the principal axes of stress. Thus, if $\sigma_1 > \sigma_2 > \sigma_3$, then $\mathbf{n}$ is given by

$$\mathbf{n} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

If the elasticity is assumed isotropic, with shear modulus G, then we obtain $|\mathbf{C}^{1/2}\mathbf{r}| = \sqrt{(2G)}\mathbf{r}$ and $|\mathbf{C}^{1/2}\mathbf{n}| = \sqrt{(2G)}\mathbf{n}$, since both $\mathbf{r}$ and $\mathbf{n}$ are purely deviatoric tensors; thus

$$\mathbf{n}^T\mathbf{C}\mathbf{r} = \frac{1}{\sqrt{2}}\frac{s_1 - s_3}{\sqrt{(2J_2)}} = \frac{\tau_{max}}{\sqrt{J_2}}$$

and hence

$$H_0 = 2G\left(1 - \frac{\tau_{max}}{\sqrt{J_2}}\right).$$

While this is zero in simple shear, it approaches $2G(1 - \sqrt{3}/2)$ for states close to uniaxial stress. It should be noted that the actual hardening modulus in simple shear, $d\tau/d\gamma^p$, is just $\frac{1}{2}H$.

If, however, two of the principal stresses are identically equal, say $\sigma_2 = \sigma_3$, then, by symmetry

$$\mathbf{n} = \sqrt{\left(\frac{2}{3}\right)}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

and it can easily be deduced that $H_0 = 0$. The actual hardening modulus $d\sigma/d\varepsilon^p$ in a uniaxial test (simple tension or compression) is $\frac{2}{3}H$.

(2) An important non-associative material model, frequently used for non-dilatant rocks and soils, uses the Lévy–Mises flow law, with $\mathbf{r}$ given by eqn (12), together with the Mises–Schleicher loading surface, which in classical plasticity coincides with the yield surface described by the equation

$$f(\boldsymbol{\sigma}, \xi) = \sqrt{J_2} - k(p, \xi) = 0$$

where $\mathbf{s}$ denotes the stress deviator, $J_2 = \frac{1}{2}\mathbf{s}^T\mathbf{s} = \frac{1}{2}s_{ij}s_{ij}$, and $p = -\frac{1}{3}\sigma_{kk}$; this yield criterion is usually called *Drucker–Prager* if k is a straight-line function of p. It follows that

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{s_{ij}}{2\sqrt{J_2}} + \frac{\mu}{\sqrt{6}}\delta_{ij}$$

where

$$\mu = \sqrt{\left(\frac{2}{3}\right)}\frac{\partial k}{\partial p}$$

may be regarded as the incremental coefficient of friction (tangent of the angle of internal friction) on the octahedral planes. Consequently

$$n_{ij} = \frac{1}{\sqrt{(1+\mu^2)}}\left(\frac{s_{ij}}{\sqrt{(2J_2)}} + \frac{\mu}{\sqrt{3}}\delta_{ij}\right).$$

For this model we can likewise use eqn (6) to determine the actual hardening moduli in specific tests, obtaining

$$\frac{\sqrt{(1+\mu^2)}}{2}H \quad \text{and} \quad \frac{3\sqrt{(1+\mu^2)}}{2+\sqrt{2}\mu}H$$

in simple shear and tension/compression, respectively.

Again assuming isotropic linear elasticity with shear modulus G and Poisson's ratio ν, we have

$$|\mathbf{C}^{1/2}\mathbf{r}| = \sqrt{(2G)}, \qquad |\mathbf{C}^{1/2}\mathbf{n}| = \frac{\sqrt{(2G)}}{\sqrt{(1+\mu^2)}}\sqrt{\left(1 + \frac{1+\nu}{1-2\nu}\mu^2\right)}, \qquad \mathbf{n}^{\mathrm{T}}\mathbf{C}\mathbf{r} = \frac{2G}{\sqrt{(1+\mu^2)}}$$

so that

$$H_0 = \frac{2G}{\sqrt{(1+\mu^2)}}\left(\sqrt{\left(1 + \frac{1+\nu}{1-2\nu}\mu^2\right)} - 1\right).$$

Note that this is independent of the state of stress, except possibly through μ.

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